LIE ALGEBRAS THAT CAN BE WRITTEN AS THE SUM OF TWO NILPOTENT SUBALGEBRAS

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This is a short survey about the current state of affairs with Lie algebras that can be written as the sum of two nilpotent subalgebras. The sum is understood in the sense of vector spaces and is not (necessarily) direct.

Motivated by similar questions from Group Theory and Ring Theory, Kegel [Ke] asked in 1963 whether a Lie ring that can be written as the sum of two nilpotent subrings, is solvable.

If we restrict our attention to algebras over a field, it is easy to see that without loss of generality one can assume that the ground field is algebraically closed. If, further, we restrict our attention to the finite-dimensional algebras, the existing powerful arsenal of structure theory immediately yields a positive answer in the zero characteristic case (see e.g. [G]). As it always happens, in characterisic p the situation becomes more complicated, and the attention to this question was renewed in 1982 by Kostrikin [Ko].

After that, Petravchuk [Pe1] gave an example providing a negative answer to the question in characteristic 2. In positive direction, a few particular results (with restrictions on the index of nilpotency of one of the summands) were obtained, until the beginning of 1990s when Panyukov [Pa1] (for characteristic $p > 2$) and myself [Z] (for characteristic $p > 5$), using different approaches, provided a positive answer in the general case.

Note that the further question arises of precise description of a class of such algebras inside the class of all solvable algebras (see [Z], [Pa2], [Pe3] and [BTT]). However, what is primarily interesting to us here is the following

Question. Is it true, that an infinite-dimensional Lie algebra that can be written as the sum of two nilpotent subalgebras, is solvable?

This was (re)asked, particulary, in [BTT] and by Rutwig Campoamor.

In [Pe2], the positive answer to this question is provided for the two cases: when one of the summands is finite-dimensional, and when

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commutants of both summands are finite-dimensional, and in [HS] - for the class of locally-finite Lie algebras (i.e. Lie algebras all whose finitely-generated algebras are finite-dimensional). Both results are obtained by a quick and easy reduction to the finite-dimensional case.

A weaker question, whether the sum of two Lie-nilpotent associative algebras is Lie-solvable, is also open. An affirmative answer is known in the case when one of the summands is one-sided ideal ([LP, Corollary 1]).

The similar results for infinite groups (with, again, that or another finitness conditions) were obtained by N. S. Chernikov (see [C] and references in [Pe2]). Also, the numerous results about finite groups that can be decomposed into a product of two groups with different properties, may be found in [F] and [H, Chap. VI], and references therein.

References

[BTT] Y. Bahturin, M. Tvalavadze and T. Tvalavadze, *Sums of simple and nilpotent Lie algebras*, Comm. Algebra **30** (2002), No.9, 4455–4471.

[C] N.S. Chernikov, *Infinite groups that are products of nilpotent subgroups*, Soviet Math. Dokl. **21** (1980), No.3, 701–703.

[F] E. Fisman, *Finite factorizable groups*, PhD Thesis, Bar-Ilan University, 1982.

[G] M. Goto, *Note on a characterization of solvable Lie algebras*, J. Sci. Hiroshima Univ. Ser. A-I **26** (1962), No.1, 1–2.

[HS] M. Honda and T. Sakamoto, *Lie algebras represented as a sum of two subalgebras*, Math. J. Okayama Univ. **42** (2000), 73–81.

[H] B. Huppert, *Endliche Gruppen. I*, Springer, 1967.

[Ke] O.H. Kegel, *Zur Nilpotenz gewisser assoziativer Ringe*, Math. Ann. **149** (1963), 258–260.

[Ko] A.I. Kostrikin, *A solvability criterion for a finite-dimensional Lie algebra*, Moscow Univ. Math. Bull. **37** (1982), No.2, 21–26.

[LP] V.S Luchko and A.P. Petravchuk, *On one-sided Lie nilpotent ideals of associative rings*, `arXiv:0803.0968`.

[Pa1] V.V. Panyukov, *On the solvability of Lie algebras of positive characteristic*, Russ. Math. Surv. **45** (1990), No.4, 181–182.

[Pa2] V.V. Panyukov, *On solvable Lie algebras decomposable into a sum of two nilpotent subalgebras*, Math. Sbornik **83** (1995), No.1, 221-235.

[Pe1] A.P. Petravchuk, *Lie algebras which can be decomposed into the sum of an abelian subalgebra and a nilpotent subalgebra*, Ukrainian Math. J. **40** (1988), No.3, 331-334.

[Pe2] A.P. Petravchuk, *On a sum of two Lie algebras with finite dimensional commutant*, Ukrainian Math. J. **47** (1995), No.8, 1244–1252.

[Pe3] A.P. Petravchuk, *On derived length of the sum of two nilpotent Lie algebras*, Visn. Mat. Mekh. Kyïv. Univ. 2001, No.6, 53–56 (in Ukrainian).

[Z] P. Zusmanovich, *A Lie algebra that can be written as a sum of two nilpotent subalgebras, is solvable*, Math. Notes **50** (1991), 909–912.